

A Schur-Block Obstruction to Uniform Real-Axis Invertibility for a Balanced Clifford-Einstein Pencil

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Abstract

We study an explicitly defined mass-parameter pencil on vector-valued elements of a complex Euclidean Clifford algebra. The pencil combines a balanced commutator/Jordan product, an Einstein-like scalar return term, a scalar shift, and Clifford reversion. Although the ambient symbol acts on a space of dimension $d2^{d+1}$, a two-dimensional Clifford block gives an exact obstruction to uniform invertibility on the real mass axis. If d is the transverse dimension, η is the scalar-return coefficient, and $|\kappa| < |m|$, then the reduced determinant is

$$\det \mathcal{S}_{\kappa, m, t}^{(\eta)} = \frac{m^2 - \kappa^2 + (1 - \eta^2 d^2)t^2}{m^2 - \kappa^2 + t^2}.$$

When $|\eta|d > 1$, the full pencil has a nonzero kernel at the finite real shell

$$t_*^2 = \frac{m^2 - \kappa^2}{\eta^2 d^2 - 1}.$$

At $|\eta|d = 1$, the same block loses uniform invertibility asymptotically. The obstruction persists on the two checkerboard real forms defined by anti-linear Clifford conjugation. We also classify the universal two-sided products that preserve a unitary real form and explain why the balanced product is the simplest interior mixture of commutator and Jordan channels. The theorem concerns only the frozen symbol defined here. It is not a classification of Clifford-valued gravitational operators, nor does it establish failure of every sectorial notion of parameter ellipticity.

Keywords. Clifford algebra; reversion; Schur reduction; parameter-dependent symbol; generalized gradient; mass shell.

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1 Introduction

Natural first-order operators on spin and Clifford bundles are commonly assembled from covariant differentiation, Clifford multiplication, and equivariant projections. Their principal symbols are often analyzed using the representation theory of generalized gradients and Stein–Weiss operators; see Branson [4] and Pilca [8]. Parameter-dependent elliptic theory asks a stronger question: whether a symbol remains uniformly invertible when a lower-order parameter is introduced, usually in a prescribed complex sector and with estimates uniform in the joint covariable and parameter [3, 6].

This paper studies a narrower finite-dimensional problem. We define a specific Clifford-valued real-axis pencil

$$\mathcal{P}_{\kappa, m}^{(\eta)}(t) = tD + \kappa I + m\tau - \eta t G\Gamma$$

on $V_{\perp}^* \otimes \text{Cl}_{d+1, \mathbb{C}}$. Here D is induced by a balanced commutator/Jordan product with a distinguished Clifford vector, τ is reversion, and $G\Gamma$ is an Einstein-like scalar return term. The central question is whether the scalar and reversion masses can make this pencil uniformly invertible for all real t .

The answer is negative whenever $|\eta|d \geq 1$. More precisely, a two-dimensional return block yields a finite real shell when $|\eta|d > 1$, and asymptotic degeneration at the threshold. The proof needs neither a full matrix realization nor the determinant of the ambient $d2^{d+1}$ -dimensional symbol. It uses four elementary Clifford identities and one 2×2 determinant.

The calculation was motivated by the Einstein-like “Shiab” contraction displayed in the 2021 *Geometric Unity* working draft [11]. That draft describes a family of possible constructions and does not uniquely specify the product or real structure used below. Accordingly, the paper is self-contained and makes no claim about every Shiab operator or Geometric Unity as a whole. The complexification issue emphasized by Nguyen and Polya [7] is logically distinct from the mass-shell obstruction proved here.

Contributions

The main contributions are:

- (i) a precise classification of universal two-sided matrix products preserving a unitary real form;
- (ii) an explicit balanced Clifford-valued pencil with clearly stated conventions;
- (iii) an exact Schur reduction to the invariant block $\text{span}_{\mathbb{C}}\{1, \gamma_0\}$;
- (iv) the closed determinant formula and shell threshold $|\eta|d \geq 1$;
- (v) an explicit nonzero kernel vector for the full transverse pencil;
- (vi) a checkerboard real-structure lemma showing that the shell is not a complexification artifact;
- (vii) three independent verification routes: exact symbolic algebra, explicit 128×128 gamma matrices in dimension fourteen, and a full Clifford-monomial real-structure check.

Claim boundary

The result proves failure of uniform invertibility on the real mass axis for the frozen family defined in equation (10). It does not:

- classify all equivariant bilinear Clifford products;
- determine every invariant block of the full pencil when $|\eta|d < 1$;
- analyze arbitrary curved backgrounds or variable coefficients;
- establish failure of every complex-sector parameter-ellipticity condition;
- refute any operator not explicitly identified with equation (10).

Proof architecture

The argument has a deliberately short dependency chain. First, the balanced product gives four local identities on the Clifford elements

$$1, \quad \gamma_0, \quad \gamma_i, \quad \gamma_0\gamma_i.$$

Second, the leading scalar/reversion block is inverted only on the two-dimensional spaces

$$E_i = \text{span}_{\mathbb{C}}\{\gamma_i, \gamma_0\gamma_i\}.$$

Third, the return map through G and Γ produces a 2×2 operator on

$$E_0 = \text{span}_{\mathbb{C}}\{1, \gamma_0\}.$$

The determinant of that operator proves the theorem. No full Clifford decomposition, numerical diagonalization, or global inverse of the leading block is used.

2 Clifford conventions and involutions

Let $V = \mathbb{R}^{d+1}$ be Euclidean with orthonormal basis $e_0, e_1, \dots, e_d$. Its complex Clifford algebra is

$$\mathcal{A} = \text{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C} = \text{Cl}_{d+1, \mathbb{C}},$$

with generators satisfying

$$\gamma_a \gamma_b + \gamma_b \gamma_a = 2\delta_{ab}. \quad (1)$$

We fix the covector $\xi = e^0$ and write

$$V = \mathbb{R}e_0 \oplus V_{\perp}, \quad V_{\perp} = \text{span}\{e_1, \dots, e_d\}.$$

The symbol space is

$$\mathcal{E} = V_{\perp}^* \otimes \mathcal{A}.$$

The grade involution α and reversion τ are defined on a homogeneous Clifford monomial $X_r = \gamma_{a_1} \cdots \gamma_{a_r}$ by

$$\alpha(X_r) = (-1)^r X_r, \quad \tau(X_r) = \gamma_{a_r} \cdots \gamma_{a_1} = (-1)^{r(r-1)/2} X_r.$$

Both extend complex linearly. Reversion is an involutive anti-automorphism:

$$\tau(XY) = \tau(Y)\tau(X), \quad \tau^2 = I.$$

Standard references for Clifford involutions and spin geometry include Lawson–Michelsohn [5], Porteous [9], and Shirokov [10].

We use the anti-linear Clifford conjugation

$$\rho = \overline{(\cdot)} \circ \alpha \circ \tau. \quad (2)$$

On a homogeneous grade- r element,

$$\rho(X_r) = (-1)^{r(r+1)/2} \overline{X_r}. \quad (3)$$

The fixed and anti-fixed real vector spaces are

$$\mathfrak{g}_+ = \{X : \rho(X) = X\}, \quad \mathfrak{g}_- = \{X : \rho(X) = -X\}.$$

They are the two checkerboard real forms relevant to the frozen model. Identifying either abstract real form with a named matrix Lie algebra requires a matrix representation and compatible Hermitian form; no such identification is needed for the shell theorem.

3 Real-form-preserving two-sided products

Let $\sharp$ be a conjugate-linear anti-involution on a full matrix realization of $\mathcal{A}$, and define

$$\mathfrak{u}_\sharp = \{X : X^\sharp = -X\}.$$

Consider only universal two-sided products built from the two associative orders,

$$m_{\alpha,\beta}(A, B) = \alpha AB + \beta BA. \quad (4)$$

Here “universal” means that the same complex coefficients are used for all inputs in a full matrix algebra of size at least two. This hypothesis deliberately excludes trace corrections, transpose-dependent terms, degree-dependent projectors, and other nonuniversal operations.

Proposition 3.1 (Classification within the two-sided class). *The product equation (4) maps $\mathfrak{u}_\sharp \times \mathfrak{u}_\sharp$ to $\mathfrak{u}_\sharp$ for all inputs if and only if*

$$\beta = -\bar{\alpha}.$$

Equivalently, there are real x, y such that

$$m_{x,y}(A, B) = x[A, B] + iy(AB + BA). \quad (5)$$

Every such product is equivariant under simultaneous conjugation.

Proof. For $A^\sharp = -A$ and $B^\sharp = -B$,

$$(AB)^\sharp = BA,$$

so

$$m_{\alpha,\beta}(A, B)^\sharp = \bar{\alpha}BA + \bar{\beta}AB.$$

Requiring this to equal $-m_{\alpha,\beta}(A, B)$ for every pair gives

$$\bar{\beta} = -\alpha, \quad \bar{\alpha} = -\beta.$$

In a full matrix algebra of size at least two, one may choose matrix units for which AB and BA are independently nonzero, so these coefficient conditions are necessary. Writing $\alpha = x + iy$ yields equation (5). Conjugation equivariance follows from associativity. $\square$

For a unit Clifford vector q , split $\mathcal{A}$ into the subspaces commuting and anticommuting with q . A direct calculation gives

$$[q, X] = 0 \implies m_{x,y}(q, X) = 2iy qX, \quad (6)$$

$$\{q, X\} = 0 \implies m_{x,y}(q, X) = 2x qX. \quad (7)$$

Thus both x and y must be nonzero if multiplication by q is to act invertibly on both elementary parity sectors. The symmetric normalization $|x| = |y|$ maximizes the smaller of the two branch magnitudes at fixed $x^2 + y^2$. We therefore freeze the balanced product

$$m_\star(A, B) = \frac{1}{2} ([A, B] + i(AB + BA)). \quad (8)$$

This is a convenient interior point of the admissible two-sided family, not a uniqueness claim.

4 The balanced Clifford-Einstein pencil

Define the algebra maps

$$D_0 = m_\star(\gamma_0, \cdot), \quad G_i = m_\star(\gamma_i, \cdot), \quad \Gamma_i = m_\star(\gamma_0 \gamma_i, \cdot). \quad (9)$$

The map D acts componentwise on $\mathcal{E}$:

$$(Dv)_i = D_0(v_i).$$

Define

$$G : \mathcal{A} \longrightarrow \mathcal{E}, \quad G(s)_i = G_i(s),$$

and

$$\Gamma : \mathcal{E} \longrightarrow \mathcal{A}, \quad \Gamma(v) = \sum_{i=1}^d \Gamma_i(v_i).$$

Reversion also acts componentwise on $\mathcal{E}$.

For real parameters t, κ, m, η , define

$$\boxed{\mathcal{P}_{\kappa, m}^{(\eta)}(t) = tD + \kappa I + m\tau - \eta t G \Gamma.} \quad (10)$$

We impose the open mass-cone condition

$$m \neq 0, \quad |\kappa| < |m|. \quad (11)$$

The ratio t/m is the real mass parameter. The notation is finite-dimensional: no global manifold, boundary condition, or sectorial complex parameter is assumed.

4.1 Motivating Einstein-like form

The structure of equation (10) is motivated by a Ricci-like contraction corrected by a scalar-return channel. In a flat local model one may regard D as the direct contraction by a distinguished Clifford vector and $G\Gamma$ as the rank-at-most- 2^{d+1} return through a single Clifford-algebra copy. In the motivating fourteen-dimensional normalization, the standard bivector convention

$$\gamma_{ab} = \frac{1}{2}[\gamma_a, \gamma_b]$$

motivates the numerical specialization

$$\eta = \frac{1}{4}, \quad d = 13.$$

The theorem treats η as free and does not require the source to determine this value uniquely; the parameterized statement makes normalization dependence explicit.

4.2 Uniform real-axis invertibility

For fixed κ, m, η , we say that the pencil is uniformly invertible on the real axis if

$$\inf_{t \in \mathbb{R}} s_{\min} \left(\mathcal{P}_{\kappa, m}^{(\eta)}(t) \right) > 0,$$

where $s_{\min}$ denotes the smallest singular value in any fixed Hermitian matrix realization. A nonzero kernel at a real t disproves this property. At the threshold, a singular value tending to zero as $|t/m| \rightarrow \infty$ also disproves uniform invertibility even without a finite shell.

5 The invariant Schur block

Set

$$E_0 = \text{span}_{\mathbb{C}}\{1, \gamma_0\}, \quad E_i = \text{span}_{\mathbb{C}}\{\gamma_i, \gamma_0\gamma_i\} \quad (1 \leq i \leq d). \quad (12)$$

The map G sends E_0 into the direct sum of the E_i , the leading block $tD + \kappa I + m\tau$ preserves each E_i , and Γ returns this image to E_0 . The following four identities contain the whole obstruction.

Lemma 5.1 (Local Clifford identities). *For each $1 \leq i \leq d$,*

$$G_i(1) = i\gamma_i, \quad G_i(\gamma_0) = -\gamma_0\gamma_i, \quad (13)$$

$$D_0(\gamma_i) = \gamma_0\gamma_i, \quad D_0(\gamma_0\gamma_i) = \gamma_i, \quad (14)$$

$$\tau(\gamma_i) = \gamma_i, \quad \tau(\gamma_0\gamma_i) = -\gamma_0\gamma_i, \quad (15)$$

$$\Gamma_i(\gamma_i) = \gamma_0, \quad \Gamma_i(\gamma_0\gamma_i) = -i. \quad (16)$$

Proof. All statements follow from equations (1) and (8). For example,

$$G_i(1) = \frac{1}{2}(0 + i(2\gamma_i)) = i\gamma_i.$$

Since $\gamma_i\gamma_0 = -\gamma_0\gamma_i$,

$$G_i(\gamma_0) = \frac{1}{2}(2\gamma_i\gamma_0 + 0) = -\gamma_0\gamma_i.$$

Likewise,

$$D_0(\gamma_i) = \frac{1}{2}(2\gamma_0\gamma_i + 0) = \gamma_0\gamma_i,$$

and, since $\gamma_0\gamma_i\gamma_0 = -\gamma_i$,

$$D_0(\gamma_0\gamma_i) = \frac{1}{2}(\gamma_i - (-\gamma_i)) = \gamma_i.$$

The reversion identities follow from order reversal. Finally, $(\gamma_0\gamma_i)^2 = -1$ gives the two Γ_i identities directly. $\square$

Define the leading mass block

$$B_{\kappa,m}(t) = tD + \kappa I + m\tau. \quad (17)$$

On E_i in the ordered basis $(\gamma_i, \gamma_0\gamma_i)$,

$$B_{\kappa,m}(t)|_{E_i} = \begin{pmatrix} \kappa + m & t \\ t & \kappa - m \end{pmatrix}. \quad (18)$$

Set

$$\Delta = m^2 - \kappa^2 + t^2. \quad (19)$$

Under equation (11), $\Delta > 0$ for all real t , and

$$B_{\kappa,m}(t)^{-1}|_{E_i} = \frac{1}{\Delta} \begin{pmatrix} m - \kappa & t \\ t & -(\kappa + m) \end{pmatrix}. \quad (20)$$

Only the inverse on $G(E_0) \subset \bigoplus_i E_i$ is required. No assertion about global invertibility of $B_{\kappa,m}(t)$ on all of $\mathcal{E}$ is used.

Define the reduced return map

$$\mathcal{S}_{\kappa,m,t}^{(\eta)} = I - \eta t \Gamma B_{\kappa,m}(t)^{-1} G \quad \text{on } E_0. \quad (21)$$

If $s \in \ker \mathcal{S}_{\kappa,m,t}^{(\eta)}$ and

$$v = B_{\kappa,m}(t)^{-1} G s, \quad (22)$$

then

$$\mathcal{P}_{\kappa,m}^{(\eta)}(t)v = G \mathcal{S}_{\kappa,m,t}^{(\eta)} s = 0.$$

The map $G|_{E_0}$ is injective, so $s \neq 0$ gives $v \neq 0$.

6 Main obstruction theorem

Theorem 6.1 (Schur-block mass-shell obstruction). *For the pencil equation (10) satisfying equation (11), the reduced determinant on E_0 is*

$$\det \mathcal{S}_{\kappa,m,t}^{(\eta)} = \frac{m^2 - \kappa^2 + (1 - \eta^2 d^2) t^2}{m^2 - \kappa^2 + t^2}. \quad (23)$$

Consequently:

(i) *If $|\eta|d > 1$, the full pencil has a nonzero kernel at the finite real shell*

$$t_*^2 = \frac{m^2 - \kappa^2}{\eta^2 d^2 - 1}. \quad (24)$$

(ii) *If $|\eta|d = 1$, the reduced determinant is positive for finite t but tends to zero as $|t/m| \rightarrow \infty$. The lifted block therefore gives normalized vectors on which the full pencil tends to zero, so uniform real-axis invertibility fails asymptotically.*

(iii) *If $|\eta|d < 1$, this particular block has no real zero and no asymptotic degeneration. No conclusion about the other invariant blocks follows.*

Proof. Using equations (13), (16) and (20), one obtains in the ordered basis $(1, \gamma_0)$

$$\Gamma B_{\kappa,m}(t)^{-1} G|_{E_0} = \frac{d}{\Delta} \begin{pmatrix} t & -i(\kappa + m) \\ i(m - \kappa) & -t \end{pmatrix}. \quad (25)$$

Let the inner matrix be M . It satisfies

$$\text{tr } M = 0, \quad \det M = -\Delta.$$

For a 2×2 trace-free matrix,

$$\det(I - cM) = 1 + c^2 \det M.$$

Taking $c = \eta t d / \Delta$ gives

$$\det \mathcal{S} = 1 - \frac{\eta^2 d^2 t^2}{\Delta},$$

which is equation (23). The alternatives follow from the sign of $1 - \eta^2 d^2$. In case (i), any nonzero reduced null vector lifts by equation (22) to a nonzero kernel vector of the full pencil. At the threshold, the small eigenvalue of the reduced block is $1 - |t|/\sqrt{m^2 - \kappa^2 + t^2} = O(t^{-2})$. Its lifted vector has norm $O(|t|^{-1})$, while its residual has norm $O(t^{-2})$; after normalization the full residual is $O(|t|^{-1})$. $\square$

Corollary 6.2 (Normalized variables). *Let*

$$a = \frac{\kappa}{m}, \quad u = \frac{t}{m}.$$

Then $|a| < 1$ and

$$\det \mathcal{S}_{a,u}^{(\eta)} = \frac{1 - a^2 + (1 - \eta^2 d^2) u^2}{1 - a^2 + u^2}. \quad (26)$$

If $|\eta|d > 1$,

$$u_*^2 = \frac{1 - a^2}{\eta^2 d^2 - 1}. \quad (27)$$

Corollary 6.3 (Fourteen-dimensional specialization). *For*

$$d = 13, \quad \eta = \frac{1}{4},$$

the shell is

$$t_*^2 = \frac{16(m^2 - \kappa^2)}{153}. \quad (28)$$

Thus the frozen fourteen-dimensional pencil fails uniform real-axis invertibility for every $m \neq 0$ and $|\kappa| < |m|$.

Corollary 6.4 (Failure of a uniform lower bound). *If $|\eta|d > 1$, then*

$$\inf_{t \in \mathbb{R}} s_{\min} \left(\mathcal{P}_{\kappa,m}^{(\eta)}(t) \right) = 0$$

because the pencil has an exact kernel at $t = t_*$. If $|\eta|d = 1$, the same infimum is zero by the normalized sequence constructed in the proof of theorem 6.1. Hence $|\eta|d \geq 1$ is an obstruction to every positive uniform real-axis lower bound for the frozen pencil.

7 Explicit kernel and non-gauge character

For the normalization $\eta = 1/4$, $d > 4$, and $m = 1$, define

$$a = \kappa, \quad u_* = \frac{4\sqrt{1 - a^2}}{\sqrt{d^2 - 16}}.$$

A reduced null vector is

$$s_* = 1 + \frac{4i(1 - a)}{(d + 4)u_*} \gamma_0. \quad (29)$$

Substitution into the 2×2 matrix equation (25) gives

$$\mathcal{S}_{a,u_*}^{(1/4)} s_* = 0.$$

The lifted transverse vector may be chosen componentwise as

$$v_i = \frac{i(d - 4)}{d(a + 1)} \gamma_i + \frac{4i}{du_*} \gamma_0 \gamma_i, \quad 1 \leq i \leq d. \quad (30)$$

Direct substitution yields

$$\mathcal{P}_{a,1}^{(1/4)}(u_*) v = 0.$$

The vector is nonzero because both coefficients cannot vanish in the mass cone. It is transverse by construction: its one-form index lies entirely in $V_\perp^*$. In a gauge-theoretic symbol complex, a longitudinal gauge variation is proportional to the distinguished covector $\xi = e^0$, whereas v has no e^0 component. Thus the exhibited vector is not a longitudinal gauge-symbol mode.

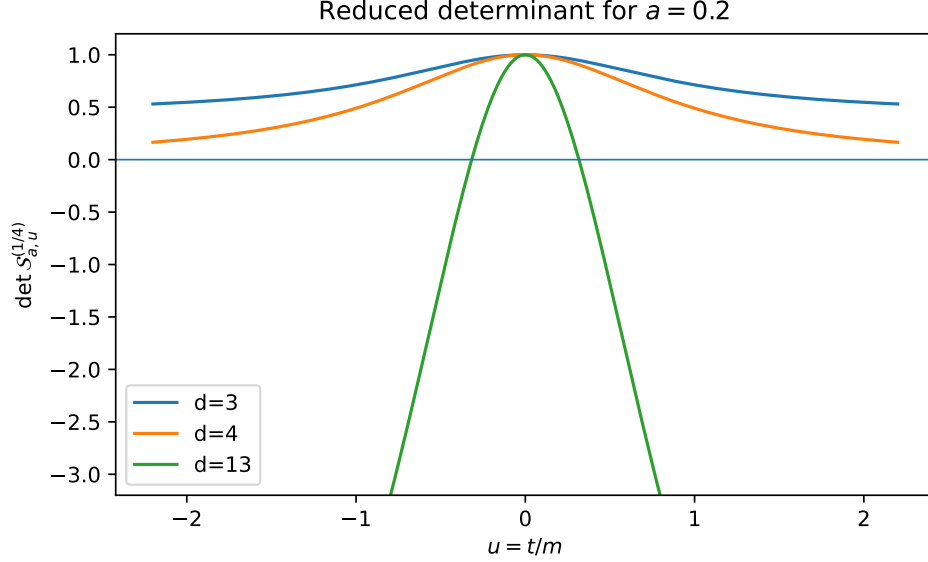


Figure 1: The normalized reduced determinant equation (26) for $a = 0.2$ and $\eta = 1/4$. For $d = 3$ the scalar block stays uniformly separated from zero; for $d = 4$ it approaches zero asymptotically; for $d = 13$ it crosses zero at a finite mass ratio. The figure illustrates only the block theorem and does not assert full invertibility in the subcritical case.

8 Checkerboard real forms

Lemma 8.1 (Compatibility of the frozen pencil with ρ). *For real t, κ, m, η , the pencil equation (10) commutes with the anti-linear involution ρ in equation (2). Hence it preserves*

$$V_{\perp}^* \otimes \mathfrak{g}_+, \quad V_{\perp}^* \otimes \mathfrak{g}_-.$$

At the shell of theorem 6.3, each real form contains a nonzero kernel vector.

Proof. For a grade-one or grade-two Clifford element A , equation (3) gives $\rho(A) = -A$. Because ρ is anti-linear and reverses products,

$$\rho([A, X]) = [A, \rho(X)]$$

and

$$\rho(i(AX + XA)) = i(A\rho(X) + \rho(X)A).$$

Therefore

$$\rho(m_{\star}(A, X)) = m_{\star}(A, \rho(X)).$$

The frozen maps D, G_i, Γ_i commute with ρ , and $\rho\tau = \tau\rho$. Thus the complete pencil commutes with ρ .

In equation (30), both grade-one and grade-two coefficients are purely imaginary. Since ρ contributes a minus sign in grades one and two and complex conjugation contributes another minus sign, $\rho(v) = v$. Hence $v \in V_{\perp}^* \otimes \mathfrak{g}_+$. By complex linearity, iv is also a kernel vector and satisfies $\rho(iv) = -iv$, so it belongs to the anti-fixed real form. $\square$

Remark 8.2. *The lemma proves that complexification does not create the shell. It does not identify $\mathfrak{g}_{\pm}$ with a particular $\mathbf{u}(p, q)$ without additional representation data.*

9 Dimension and normalization dependence

The threshold depends only on the dimensionless combination $|\eta|d$. This makes normalization dependence transparent.

Condition	Reduced behavior	Consequence
$ \eta d < 1$	bounded away from zero on this block	no obstruction from E_0
$ \eta d = 1$	tends to zero as $ t/m \rightarrow \infty$	no uniform invertibility
$ \eta d > 1$	crosses zero at finite t/m	explicit full-symbol kernel

For $\eta = 1/4$, the threshold occurs at $d = 4$. The first several dimensions are summarized below.

d	$ \eta d$	Scalar return-block status
1	1/4	no obstruction from this block
2	1/2	no obstruction from this block
3	3/4	no obstruction from this block
4	1	asymptotic degeneration
5 and above	> 1	finite real shell
13	13/4	shell in equation (28)

Changing the Fourier sign sends $t \mapsto -t$ and leaves the determinant unchanged. Changing the sign of the scalar return coefficient sends $\eta \mapsto -\eta$ and likewise leaves equation (23) unchanged. Only the magnitude of η matters for this block.

10 Relation to ellipticity and variational questions

The theorem concerns uniform invertibility of a joint real-axis symbol, not ordinary principal ellipticity. A first-order principal symbol may be invertible for every nonzero covector while the addition of a scalar parameter creates a finite shell. This is the finite-dimensional analogue of the distinction between high-frequency ellipticity and a global spectral gap.

Similarly, the theorem is independent of whether the pencil arises as the Hessian of a real action. In the broader reconstruction that motivated the model, the balanced commutator/Jordan mixture was considered because pure endpoint products have complementary algebraic kernels. A separate formal-adjoint analysis is required to decide whether an undoubled first-order bosonic action can generate the mixed equation. That variational question is not needed for the present obstruction and is intentionally left outside the theorem.

The conclusion should therefore be read as follows:

even if the balanced principal symbol is acceptable,

the canonical scalar and reversion masses do not supply a uniform real-axis gap when $|\eta|d \geq 1$.

11 Motivating specialization to the Geometric Unity draft

The 2021 working draft introduces an Einstein-like contraction of an adjoint-valued two-form using pure-trace Clifford tensors, a Ricci-like first term, and a scalar-like nested correction [11]. The draft also describes multiple possible Shiah constructions and notes that an earlier representation-theoretic

selection was not recovered. This leaves product choice, normalization, and real-form implementation underdetermined at the level required for a unique local symbol.

The frozen model in this paper makes one explicit choice:

- Euclidean complex Clifford algebra in total dimension fourteen;
- transverse dimension $d = 13$;
- the balanced product equation (8);
- reversion as the zero-order involution;
- scalar return coefficient $\eta = 1/4$ under the stated bivector normalization.

For this model, theorem 6.3 gives an exact finite shell. The result does not show that every possible completion of the draft has the same shell. It shows that this natural, explicitly defined completion does.

Nguyen and Polya argue that the public Shiab proposal faces a separate complexification problem [7]. The checkerboard lemma above addresses a different question: once the present frozen pencil is defined, its shell persists on real fixed subspaces and is therefore not removed by retreating from the complexified module.

12 Reproducibility and independent checks

The supplementary archive contains three independent implementations.

12.1 Exact symbolic reduction

A SymPy script constructs the 2×2 block from equation (25), simplifies its determinant, substitutes the predicted shell, and verifies an exact null vector. The symbolic result is

$$\det \mathcal{S}_{a,u}^{(\eta)} = \frac{1 - a^2 + (1 - d^2 \eta^2) u^2}{1 - a^2 + u^2}.$$

All tests use exact rational and symbolic arithmetic.

12.2 Explicit gamma matrices

A second script constructs fourteen Euclidean gamma matrices as tensor products of Pauli matrices, producing a 128×128 representation. It verifies the Clifford relations, the local identities, the reduced shell, and the full transverse residual. For the test value $a = 0.2$,

$$u_* = 0.3168472137525358,$$

and the output is

$$\begin{aligned} \max \|\gamma_a \gamma_b + \gamma_b \gamma_a - 2\delta_{ab} I\| &= 0, \\ s_{\min}(\mathcal{S}) &\approx 1.13 \times 10^{-16}, \\ \|\mathcal{P}v\| &\approx 2.26 \times 10^{-14}. \end{aligned}$$

12.3 Full monomial real-structure check

A third implementation represents all 2^{14} Clifford monomials by bitmasks. It checks the checkerboard signs, the commutation of reversion with ρ , compatibility of the balanced product for all relevant grade-one and grade-two fixed tensors, and the reality of the explicit shell vector. Every algebraic error count is zero.

The computations corroborate the proof but are not used as substitutes for it.

13 Minimal hand-reproduction protocol

The central result can be checked without the supplementary software. A referee need only perform the following steps.

1. Use $\gamma_0^2 = \gamma_i^2 = 1$ and $\gamma_0\gamma_i = -\gamma_i\gamma_0$ to verify equations (13) to (16).
2. Write $B_{\kappa,m}(t)|_{E_i}$ in the basis $(\gamma_i, \gamma_0\gamma_i)$ and verify equations (18) and (20).
3. Apply G to $x + y\gamma_0$, apply the inverse block componentwise, and return with Γ .
4. Confirm the matrix in equation (25) and the identities $\text{tr } M = 0$, $\det M = -\Delta$.
5. Compute $\det(I - cM) = 1 + c^2 \det M$ with $c = \eta t d / \Delta$.
6. At $|\eta|d > 1$, substitute equation (24) and lift any reduced null vector by equation (22).
7. Check theorem 8.1 to obtain fixed and anti-fixed real kernel vectors.

Every theorem-level claim in the paper depends on this chain. The larger computer checks are independent corroboration and regression tests.

14 Limitations, open cases, and falsifiability

The theorem is intentionally local and finite-dimensional. Four boundaries are especially important.

Subcritical return coefficient. When $|\eta|d < 1$, the block E_0 remains uniformly separated from zero. Other Clifford blocks may still obstruct uniform invertibility, but the present proof neither establishes nor excludes that possibility.

Other products and zero-order operators. The balanced product is one natural interior point of the two-sided family in equation (5). Degree-dependent products, transported projectors, additional bundle multiplicities, and zero-order maps other than scalar shift plus reversion lie outside the theorem.

Variable coefficients and geometry. The pencil is a frozen local symbol. Curvature, variable coefficients, boundary conditions, and global topology can create further obstructions or alter the spectral problem. They cannot remove the exact algebraic kernel of this frozen family without changing the frozen operator itself.

Mass-cone boundary and exterior. The assumption $|\kappa| < |m|$ guarantees that the only inverse used in the proof exists for every real t . The regimes $|\kappa| \geq |m|$ require a separate case analysis and are not needed for the stated no-go result.

The theorem is directly falsifiable. It fails if any one of the four identities in theorem 5.1, the inverse equation (20), or the return matrix equation (25) is incorrect under the stated conventions. The supplementary implementations test precisely these points using independent algebraic representations.

15 Prior art and novelty boundary

The nearest established bodies of work are:

- generalized gradients and ellipticity [4, 8];
- Clifford involutions, reversion, and invariant subspaces [5, 9, 1, 2, 10];
- parameter-dependent elliptic problems and uniform estimates [3, 6];
- the Geometric Unity draft and its public technical response [11, 7].

Within the bounded searches performed for this draft, we did not locate the determinant equation (23) or the threshold $|\eta|d \geq 1$ for the balanced commutator/Jordan product with reversion and scalar return. This is not proof of exhaustive priority. A specialist MathSciNet and zbMATH search remains appropriate before journal submission.

16 Conclusion

A large Clifford-valued mass-parameter symbol can fail uniform real-axis invertibility for a reason visible in a two-dimensional return block. For the balanced product, scalar correction, scalar shift, and reversion used here, the exact reduced determinant is

$$\frac{m^2 - \kappa^2 + (1 - \eta^2 d^2)t^2}{m^2 - \kappa^2 + t^2}.$$

The threshold $|\eta|d = 1$ separates a subcritical block, asymptotic degeneration, and a finite real shell. In the motivating fourteen-dimensional normalization $d = 13$, $\eta = 1/4$, the full pencil has an explicit nonzero kernel at

$$t_*^2 = \frac{16(m^2 - \kappa^2)}{153}.$$

The shell persists on both canonical checkerboard real forms. The proof is short, exact, and independent of a full gamma-matrix realization.

A Detailed calculation of the return matrix

Let

$$s = x + y\gamma_0 \in E_0.$$

Using equation (13),

$$G_i(s) = ix\gamma_i - y\gamma_0\gamma_i.$$

With the inverse equation (20), the coefficients in the basis $(\gamma_i, \gamma_0\gamma_i)$ become

$$\frac{1}{\Delta} \begin{pmatrix} m - \kappa & t \\ t & -(\kappa + m) \end{pmatrix} \begin{pmatrix} ix \\ -y \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} i(m - \kappa)x - ty \\ itx + (\kappa + m)y \end{pmatrix}.$$

Applying equation (16) and summing over i gives

$$\Gamma B^{-1}G(s) = \frac{d}{\Delta} [(tx - i(\kappa + m)y) + (i(m - \kappa)x - ty)\gamma_0],$$

which is precisely equation (25).

B Direct verification of the normalized null vector

For $\eta = 1/4$, define

$$q = \frac{4i(1 - a)}{(d + 4)u_*}, \quad s_* = 1 + q\gamma_0.$$

At

$$u_*^2 = \frac{16(1 - a^2)}{d^2 - 16},$$

the first row of $\mathcal{S}s_* = 0$ reduces to

$$1 - \frac{du_*^2}{4(1 - a^2 + u_*^2)} + \frac{idu_*(a + 1)}{4(1 - a^2 + u_*^2)}q = 0.$$

Substitution of q and the shell relation reduces the numerator to zero. The second row is dependent because the determinant vanishes and the first-row coefficient vector is nonzero. The lifted vector obtained from $B^{-1}Gs_*$ simplifies to equation (30).

C Algorithmic representations

C.1 Bitmask Clifford multiplication

Index a monomial by a bitmask $I \subseteq \{0, \dots, d\}$. Left multiplication by γ_a toggles bit a and contributes the sign

$$(-1)^{\#\{b \in I: b < a\}}.$$

Right multiplication contributes

$$(-1)^{\#\{b \in I: b > a\}}.$$

This representation uses no dense matrices and makes exact checking over the full monomial basis inexpensive.

C.2 Tensor-product gamma matrices

For even total dimension $2r$, one convenient Euclidean representation is

$$\begin{aligned}\gamma_{2j} &= \sigma_3^{\otimes j} \otimes \sigma_1 \otimes I^{\otimes(r-j-1)}, \\ \gamma_{2j+1} &= \sigma_3^{\otimes j} \otimes \sigma_2 \otimes I^{\otimes(r-j-1)},\end{aligned}$$

for $j = 0, \dots, r - 1$. For total dimension fourteen, $r = 7$ and the matrices are 128×128 .

D Low-dimensional shell table for $a = 0.2$ and $\eta = 1/4$

d	Status	$ u_* $
1	no obstruction from E_0	—
2	no obstruction from E_0	—
3	no obstruction from E_0	—
4	asymptotic degeneration	∞
5	finite shell	1.306394529484
6	finite shell	0.876356092008
7	finite shell	0.682242292338
8	finite shell	0.565685424949
9	finite shell	0.486114896200
10	finite shell	0.427617987060
11	finite shell	0.382473154987
12	finite shell	0.346410161514
13	finite shell	0.316847213753
14	finite shell	0.292118697336

E Proof-dependency map

The logical dependencies of the principal statements are:

equations (1) and (8) $\implies$ theorem 5.1 $\implies$ equations (18), (20) and (25) $\implies$ theorem 6.1.

The explicit kernel in equation (30) depends on the finite-shell branch of theorem 6.1. The real-form conclusion depends additionally on equation (2) and theorem 8.1. The classification in theorem 3.1 motivates the balanced product but is not required for the determinant calculation once equation (8) is adopted as a definition. The Geometric Unity discussion is motivational only and is not used in any proof.

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Data and code availability. Exact symbolic, explicit matrix, low-dimensional, and real-structure verification scripts accompany this release. The public-release package includes a manifest and cryptographic checksums. A repository DOI or stable URL should replace this sentence after deposit.

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Use of artificial intelligence tools. OpenAI’s ChatGPT was used as an exploratory mathematical, coding, and editorial assistant, including generation and criticism of candidate derivations, draft verification code, and manuscript organization. The author reviewed the theorem statements, executed the supplied verification scripts, and accepts responsibility for all definitions, proofs, computations, citations, and claims. The AI system is not an author.

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